Exercise 12

Solve the initial-value problem.

$$y'' - 6y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = 1$$

Solution

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} - 6(re^{rx}) + 25(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 6r + 25 = 0$$

Solve for r.

$$r = \frac{6 \pm \sqrt{36 - 4(1)(25)}}{2} = \frac{6 \pm \sqrt{-64}}{2} = 3 \pm 4i$$

$$r = \{3 - 4i, 3 + 4i\}$$

Two solutions to the ODE are $e^{(3-4i)x}$ and $e^{(3+4i)x}$. According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y(x) &= C_1 e^{(3-4i)x} + C_2 e^{(3+4i)x} \\
&= C_1 e^{3x} e^{-4ix} + C_2 e^{3x} e^{4ix} \\
&= e^{3x}(C_1 e^{-4ix} + C_2 e^{4ix}) \\
&= e^{3x}[C_1(\cos 4x - \sin 4x) + C_2(\cos 4x + i \sin 4x)] \\
&= e^{3x}[(C_1 + C_2)\cos 4x + (-iC_1 + iC_2)\sin 4x] \\
&= e^{3x}(C_3 \cos 4x + C_4 \sin 4x)
\end{aligned}$$

Differentiate it with respect to x.

$$y'(x) = 3e^{3x}(C_3 \cos 4x + C_4 \sin 4x) + e^{3x}(-4C_3 \sin 4x + 4C_4 \cos 4x)$$

Apply the initial conditions to determine C_3 and C_4.

$$y(0) = C_3 = 2$$

$$y'(0) = 3C_3 + 4C_4 = 1$$

Solve the system.

$$C_3 = 2 \qquad C_4 = -\frac{5}{4}$$

Therefore,

$$y(x) = e^{3x}\left(2\cos 4x - \frac{5}{4}\sin 4x\right).$$

Below is a plot of the solution versus x.